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FRACTIONS, REUNITISATION AND THE NUMBER-LINE REPRESENTATION

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This paper reports on a study in which Years 6 and 10 students were individually interviewed to determine their ability to unitise and reunitise number lines used to represent mixed numbers and improper fractions. Only 16.7% of the students (all Year 6) were successful on all three tasks and, in general, Year 6 students outperformed Year 8 students. The interviews revealed that the remaining students had incomplete, fragmented or non-existent structural knowledge of mixed numbers and improper fractions, and were unable to unitise or reunitise number lines. The implication for teaching is that instruction should focus on providing students with a variety of fraction representations in order to develop rich and flexible schema for all fraction types (mixed numbers, and proper and improper fractions).

In summarising the literature, Behr, Harel, Post, & Lesh (1992) claimed that there are five subconstructs of rational number, namely, *part/whole, quotient, measure, ratio number*, and *operator*, and that comprehending rational numbers means having an understanding of each subconstruct as well as their interrelatedness.

Australian mathematics syllabi focus primarily on the *part/wole* subconstruct. Under this subconstruct, a *fraction* is a generic term used to denote a numerical amount that is a *part of a whole* (Kieren, 1983; Nik Pa, 1989; Payne, Towsley, & Huinker, 1990), where the whole is any continuous quantity (e.g., a region/area, a line or a volume) or discrete quantity (e.g., a set of objects). Thus, children's ability to interpret fractions is highly dependent on their notion of a unit (whole), their ability to partition the whole (Lamon, 1996; Pothier & Sawada, 1983), and to reconstruct units (Behr et al., 1992; Lamon, 1996; Nik Pa, 1989; Steffe, 1986). According to Steffe (1986), there are four different ways of thinking about a unit, namely, *counting (*or *singleton) units*, *composite units*, *unit-of-units* and *measure units*, with each type apparently representing an increasing level of abstraction There is a consensus in the literature (Behr, Harel, Post, & Lesh, 1992; Harel & Confrey, 1994; Hiebert & Behr, 1988; Lamon, 1996) that the cognitive complexity involved in connecting representations, symbols and operations can be attributed mainly to the changes in the nature of the unit. In particular, the complexity required to process unit-of-units and measure units has major implications for acquiring an understanding of rational numbers, particularly in relation to concrete and pictorial representations.

Whatever the representation of the whole, fundamental to the *part/whole* subconstruct is the notion of *partitioning* a whole into a number of *equal* parts and composing and recomposing (i.e., *unitising* and *reunitising*) the equal parts to the initial whole. According to Kieren (1983), partitioning experiences may be as important to the development of rational number concepts as counting experiences are to the development of whole number concepts. Students, therefore, should be provided with several opportunities to partition a variety of fraction models in a variety of ways so that they come to understand that ½ (for example) always represents one of two equal pieces. *Partitioning, unitising* and *reunitising* are often the source of students' conceptual and perceptual difficulties in interpreting rational-number representations (Baturo, 1997; Behr et al, 1992; Kieren, 1983; Lamon, 1996; Pothier & Sawada, 1983). In particular, reunitising, the ability to change one's perception of the unit, requires a flexibility of thinking that may be beyond young children.

Australian syllabi advocate the use of the area model in developing the initial understanding of a fraction because of the conceptual and perceptual difficulties students have in interpreting the other models (Payne, 1976). For example, with the set model, students find it difficult to unitise a group of discrete objects (Behr et al., 1992; Nik Pa, 1989); with the linear model, children tend to see the marks as discrete points on a line instead of as parts of a whole unit and, again, the problem is related to unitising; with the volume model, the equal partitions are often not shown. Although the set, linear and volume models are not used in the initial development of the part/whole notion of fractions, they should not be avoided as full understanding of any notion requires an ability to abstract the salient features from a variety of materials (Dienes, 1969). In his study involving 220 college students, Silver (1983) reported on what he called *representational rigidity,* a limitation in the variety of mental models that was available to the students. This limitation appeared to be a major inhibitor of the students' ability to operate on fractions.

There has been a recent resurgence of interest in the number line representation, for place value (Bove, 1995), mental computation (Beishuizen, 1997), word problems (Okamoto, 1996), fractions (Maher, Martino, & Davis, 1994), percent problems (Dole, 1998; Parker & Leinhardt, 1995), and functions (Olsen, 1995). This would appear to be in conflict with the earlier literature (e.g., Payne, 1976; Payne et al., 1990) that stressed the conceptual difficulties students had in unitising and reunitising fractions represented by number lines. However, the number line appears to be an ideal representation to help students connect whole-number and fraction processes such as counting (e.g., 3 fifths, 4 fifths, 5 fifths, 6 fifths … is isomorphic to counting whole numbers). A well, the partitions on a number line showing fractional parts can be recorded as improper fractions or as mixed numbers, thus strengthening the understanding that these two forms can be used interchangeably.

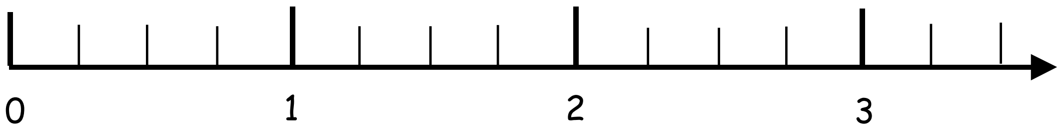
This paper reports on Years 6 and 8 students' responses to tasks involving placing a mixed number and improper fractions of a number line. The study's impetus was an interest in students' ability to use this representation, particularly with respect to unitising and reunitising, and the conflicting reports concerning number line success.

The study

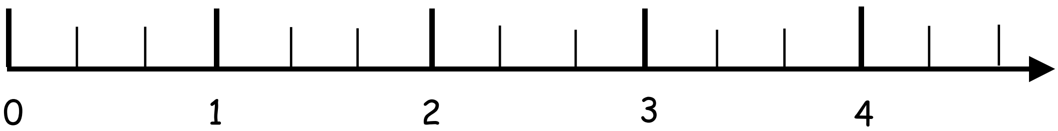
Subjects. The subjects were 24 Year 6 students (12 girls, 12 boys) from 3 suburban and 2 regional primary schools and 10 Year 8 students (5 girls, 5 boys) from 2 regional secondary schools. The schools were generally in lower middle-class areas. The students were chosen by their teachers to represent a cross-section of abilities in their classes (not including extremes).

Instrument. There were three interview tasks (see Figure 1) which were designed to represent a sequence of cognitive difficulty (simplest to hardest).

Task 1: Show $2\,{}^{1}/_{4}$ on the number line below.



Tasks 2 and 3: Show ${}^{6}/_{3}$ and ${}^{11}/_{6}$ on the number line below.



Unitisa with physical or mental repartitioning) was required for Task 3. Task 1 was included to give all students the most chance of being correct in that it required unitisation only and the whole number provided a visual clue to the particular whole to be considered on the number line. Task 2 was considered to be more difficult that Task 1 because of the improper fraction recording of a whole number. It was thought that this would be a nonprototypic representation for most students. Task 3 was considered to be more difficult than Task 2 because, apart from the nonprototypic recording, it required reunitisation.

Procedure. The students were withdrawn from class and interviewed individually. They were asked to place the numbers and then to explain their responses. Before doing this, the tasks were read to the students to alleviate any difficulties that might arise from: (a) the use of the slash instead of the vinculum in the fraction recording, and (b) an inability to read fractions. The interviews were audiotaped.

Analysis. The students' responses and reasons were recorded and categorised into commonalities. Inferences were drawn with respect to the knowledge and strategies held by students giving certain responses.

Results

The results for each task were categorised in terms of correctness and closeness to correctness (see Table 1).

Task 1. When asked to explain their responses, students with correct placement tended to respond in terms of the existing parts on the number line and to acknowledge that the 2 and the ${}^{1}/_{4}$ were separate [e.g., *because they're all quarters, and that's two* (the whole number) *and that's a quarter*].

Table 1
Number (%) of Task Response Categories for All Tasks by Year Level.

	Students	
Task response category	**Year 6 (*n*=24)**	**Year 8 (*n*=10)**
Task 1: $2^1/_4$		
Correct ($2^1/_4$)	54.2	60.0
Other quarters ($2^2/_4$, $2^3/_4$)	16.7	20.0
Eighths ($2^1/_8$, $2^5/_8$, $2^7/_8$)	20.8	20.0
Could not do	08.3	00.0
Task 2: $^6/_3$		
Correct ($^6/_3$)	25.0	10.0
Near ($1^5/_6$, $1^2/_3$, $2^1/_3$, $2^2/_3$)	20.8	20.0
High ($3^5/_6$, 4, $6^1/_3$, $6^2/_3$, $6^3/_4$)	29.2	10.0
Low ($^1/_6$, $^1/_2$, 1)	00.0	30.0
Could not do	25.0	30.0
Task 3: $^{11}/_6$		
Correct ($^{11}/_6$)	25.0	00.0
Near (2, $1^1/_3$, $1^2/3$)	08.3	20.0
Mid ($3^1/_3$, $3^2/_3$, $3^5/_6$, 4)	20.8	30.0
High ($11^1/_3$, $11^1/_2$)	08.3	00.0
Low ($^1/_2$)	00.0	10.0
Could not do	37.5	40.0

The responses from the 6 students who marked incorrectly but focused on quarters (i.e., $2^2/_4$ or $2^3/_4$) were idiosyncratic. For example, of the 2 students who marked $2^2/_4$, one indicated prototypic thinking in that she seemed to associate the unit numerator with a half (*it has 1 here so it's in the middle*); the other student seemed to combine $2^1/_4$ with another quarter (*there's two up there and it goes up in quarters ... goes one quarter, two quarters, three quarters ... two quarters, so there)*. Of the 4 students who marked $2^3/_4$, three seemed to associate the unit numerator with meaning 1 "space" less than the next whole number whilst the remaining student seemed to interpret $2^1/_4$ as having 2 wholes, 2 quarters, and then add on the ¼ [*cos there's two ... zero to two* (pointing at whole numbers) ... *and you go up four* (pointing to quarters) ... *go to two* (quarters) *and about there* ($^3/_4$)]. The responses from students who marked eighths generally appeared to ignore the relationship between the denominator and the number of partitions. They seemed to have developed a holistic view of mixed numbers so that $2^1/_4$ was *just a little past 2* (marking $2^1/_8$) or *just a bit before 3* (marking $2^7/_8$). The student who marked $2^5/_8$ indicated that he also associated the unit numerator with a half (*cos that's 2 and a quarter*) and then marked just past the ½.

The students who could not place the number were generally confused by the fourths. They counted the marks and, therefore, saw the intervals as thirds [e.g., *two* (pointing to the whole number) *and then it goes one, two, three instead of four* (pointing to the partitions)].

Task 2. Unlike Task 1, the Year 6 students' responses were more accurate than those of the Year 8 students. Students with correct placement tended to invoke the quotient (*2 threes are 6*) or operator (*6 thirds is 2*) subconstructs as well as the part/whole subconstruct (i.e., counting in thirds). Although correct, 2 of the latter students (part/whole) were unsure of their answers, for example, *I'm pretty confused about this but with thirds, there's 3 in each one of them* (the units) *but, if you count them like that, there's 6 there* (at 2). It seems as though this student had had little or no experience with renaming whole numbers (with the possible exception of 1) as fractions and that this was a nonprototypic task for her. The other student was confused between the 4 intervals (and therefore quarters) and the three partitions (and therefore thirds).

The explanations from students who marked "near" the correct response (e.g., $1^5/_6$, $1^2/_3$, $2^1/_3$, $2^2/_3$) showed that they tended to count thirds but either counted the partition lines starting at 0 ($1^2/_3$ or $1^5/_6$ – $1^2/_3$ *with a little bit on*), counted an extra third (*I went 1, 2, 3, 4 5, 6 and then a third more* – $2^1/_3$), or thought that the partitions at the whole numbers could not be counted because they were not the same size as the internal partitions ($2^2/_3$). The responses from students who marked "high" (i.e., beyond 3) indicated that they thought the 6 in $^6/_3$ was a whole number. This belief was so strong that the students extended the number line to past the whole number 6 so that they could position the improper fraction [e.g., *six and a bit more* – marking $6^1/_3$; *you've got to put thirds* (between 6 and 7) *so you'd mark the 3rd one* – counting the partition at 6 as the first third and thus marking $6^2/_3$; *six, and three partitions for the third* – marking $6^3/_4$]. The responses from students who marked "low" (i.e., $^1/_6$, $^1/_2$, 1) showed very poor understanding of improper fractions and confusion between representations (e.g., *it's six threes ... six threes are eighteen ... so 18% of the whole* – marking $^1/_6$; *it's half* – interpreting the fraction as $^3/_6$; *I counted up six in a row and came back three* – marked 1).

Students who could not place the number tended to have the same misconception as those marking high (but without the initiative to extend the line), namely, equating the numerator with the whole number, 6 (e.g., *there's no six*; *not enough numbers*). One student couldn't interpret the improper fraction because it didn't have a whole number before it and therefore, there was no "range marker" [e.g., *there's 3 here* (counting the whole numbers) *but it didn't say a whole number to put it between, or, after or before*].

Task 3. As for Task 2, the Year 6 students were more accurate than the Year 8 (none of whom gave the correct response). Except in one instance, correct placement involved either overt physical repartitioning or mental repartitioning of thirds into sixths (e.g., *well there's only thirds ... but if you count two of them, there'd kind of like be sixes - 1 sixth, 2 sixths*; *cut each third in halves*; *you'd have to draw them all*

... put one in between). The exception was the boy who invoked the quotient subconstruct (as he had for Task 2).

The "near" placements at $1^1/_3$ and $1^2/_3$ were based on idiosyncratic strategies, neither of which involved reunitising thirds as sixths [e.g., $^{11}/_6$ was interpreted as $1^1/_6$ and the partition after 1 was marked – $1^1/_3$; *2 sixes are 12 so take off 1* (third) – $1^2/_3$]. The "near" placement at 2 revealed erroneous thinking (e.g., $^1/_6 - ^1/_3 + ^1/_3$; $^1/_6 > ^1/_3$). Generally, the "mid" errors were based on either counting the thirds as sixths and consequently marking $3^2/_3$ (if parts were counted correctly) or $3^1/_3$ or 4 (if partition lines, not intervals, were counted). "High" placement at $11^1/_3$ and $11^1/_2$ resulted from associating the numerator with whole numbers on the number line and then partitioning the unit from 11 to 12 into thirds. Again, this belief was so strong that the students extended the number line to past the whole number 11. The "low" placement at $^1/_2$ was a consequence of the same confusion between percent and fractions as for Task 2

The students who could not place the number appeared not to comprehend the fraction, the number line or how the two could go together (e.g., *you can't get it on there* and *but that's not sixths*). They exhibited no confidence and little interest in attempting the problem.

Discussion

The poor results for the simplest task (Task 1) suggests that there is an inherent cognitive difficulty involved in conceptualising the number line representation of fractions. The difficulty appears to be compounded when the given fraction is recorded in improper fraction form. These arguments are supported by the increased number of students who could not attempt to answer Tasks 2 and 3 (see Table 2).

Incorrect responses were idiosyncratic, that is, few error patterns were discerned, and same responses nearly always were the result of different (and inappropriate) thinking strategies. Those error patterns that were discerned were: (a) counting partition lines rather than intervals and therefore counting included 0; (b) associating the numerator (in improper fractions) with a whole-number marker and therefore counting whole numbers rather than parts; (c) failure to reunitise when required because of lack of awareness (and therefore a metacognitive shortcoming) or lack of knowledge (e.g., 1 sixth is composed by doubling a third, rather than halving a third).

Furthermore, maturation appears to have no effect on performance. In fact, performance in this study dropped with age (particularly for the improper fractions), a phenomenon we found difficult to understand, particularly as we had expected the Year 8 students to have been much more exposed to this form of fraction recording in view of the fact that they would have encountered addition and subtraction of unlike common fractions requiring decomposition, as well as being exposed to percent conversions to decimal and common fractions, and to prealgebra tasks. We tentatively suggest that, if appropriate structural knowledge has not been constructed (i.e., semantic knowledge), then students are forced to create "rules", the number of which increase as more and more knowledge needs to be accommodated. The result of this would be to have no means of solving nonprototypic tasks or to invoke as many "rules" as one can think of. One student (Year 8) exemplifies this latter situation as his protocol shows. In Task 2, he had placed $^{6}/_{3}$ (which he read as 6 threes) at $^{1}/_{6}$ and, in Task 3, placed $^{11}/_{6}$ at $^{1}/_{2}$.

Task 2: *Six threes are 18 so 18% of the whole* (0 to 1) – *about there* ($^{1}/_{6}$).

Task 3: *I timesed 6 by 11 – 66% in a hundred so I took it to the nearest part, point five* ($^{1}/_{2}$).

Conclusions

This study found that students have conceptual difficulties in placing proper (e.g., $^{1}/_{4}$ in $2^{1}/_{4}$) and improper fractions on number lines may have been partitioned into the appropriate number of parts. Thus, the results tend to support Payne's (1976) findings that students have difficulties with unitising units on a number line. In particular, there appears to be confusion between whole and part [e.g., not counting the wholes ($^{6}/_{3}$ is placed at $2^{2}/_{3}$), counting markings not spaces ($^{6}/_{3}$ is placed at $1^{2}/_{3}$), and counting wholes as parts ($^{6}/_{3}$ is placed somewhere after 6)].

The study also found that the placement problems were exacerbated when the number of partitions did not match the given denominator, indicating a continuing difficulty with partitioning and unitising/reunitising on a number line (e.g., counting thirds instead of sixths so that $^{11}/_{6}$ is placed at $3^{2}/_{3}$; not knowing how to reunitise thirds as sixths). This finding reinforces the consensus in the literature on the fundamental importance of children's notion of the unit with respect to representations (e.g., Behr et al., 1992; Harel & Confrey, 1994; Kieren, 1983; Lamon, 1996; Pothier & Sawada, 1983).

The major implications of the findings are that: (a) unless teachers are aware of the inherent conceptual problems students have in processing number lines, their effective use as a teaching and problem solving aid will be limited; and (b) further research which focuses on analyses of students' comprehension of number line fraction representations is required in view of the current resurgence of interest in the area (Beishuizen, 1997; Bove, 1995; Okamoto, 1996; Dole, 1998; Olsen, 1995). An exhaustive review of the PME proceedings dating back to 1994 produced very few articles in this area, supporting the need for further research.

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